Part A - Multiple-choice Questions [40 marks]

Question A.1 [3 marks]

Define an *unfortunate number* as a number which is 13 times the sum of its digits. How many unfortunate numbers are there?

(A) None

(B) 3

(C) 5

(D) Infinitely many

Solution: It is clear there are no solutions with one digit. No solution can have 4 or more digits since, for example, any 4 digit number is bounded below by 1000, but 13 times its digit sum is bounded above by $13 \times 4 \times 9 = 468$.
Considering two digit possibilities gives no solutions. Now for 3 digit numbers abc we need $100a + 10b + c = 13(a + b + c)$, so $29a = b + 4c$. The right-hand side is bounded above by 45 (when $b = c = 9$), which forces $a = 1$. Then we have 4 divides $29 - b$. This gives 3 values for b and 3 corresponding values for c. The answer is (B).

Question A.2 [3 marks]

Let $q \geq 0$. The minimum value of

$$I(q) = \int_0^1 (x - q)^2 dx + \int_{-q}^{q} \sin^9(x)(|x| + \cos(x) + q)dx$$

as q varies is achieved at

(A) $q = 0$

(B) $q = \frac{1}{2}$

(C) $q = 1$

(D) Multiple values of q

Solution: The second integral is an odd function integrated over a symmetric range; the second integral vanishes. The first integral can be computed as $I(q) = \frac{1}{3} - q + q^2 = \left(q - \frac{1}{2}\right)^2 + \frac{1}{12}$, minimised at $q = \frac{1}{2}$. The answer is (B)

Question A.3 [3 marks]

How many integers n are there such that $\frac{n}{100-n}$ is also an integer?

(A) 12

(B) 15

(C) 18

(D) 21

Solution: We have $\frac{n}{100-n} = \frac{100}{100-n} - 1$. We need n such that $100 - n$ divides 100. One example is $n = 0$. Listing the factors of 100 and matching n accordingly leads to the list $\{0, 50, 75, 80, 90, 95, 96, 98, 99, 101, 102, 104, 105, 110, 120, 125, 150, 200\}$, which has 18 elements; the answer is (C).

Question A.4 [5 marks]

Assume angles are given in degrees. The sum

$$S = \cos^2(1) + \cos^2(2) + \cdots + \cos^2(270)$$

equals

(A) 134

(B) 134.5

(C) 135

(D) 135.5

Solution: Note that $\cos(90 - x) = \sin(x)$ and $\cos(90) = 0$. Consider the first 90 terms. We have $\sin^2(1) + \cos^2(1) + \sin^2(2) + \cos^2(2) + \cdots + \sin^2(44) + \cos^2(44)$. There is no term to pair with $\cos^2(45)$ and so this contributes another 0.5 to the sum. Noting also that $\sin(180 - x) = \sin(x)$ allows for similar calculation of the remaining terms. The sum from 91 to 180 gives 45.5 due to $\cos(180) = -1$. The sum of the final 90 terms is the same as the sum of the first 90. Putting this all together, we reach $S = 44.5 + 45.5 + 44.5 = 134.5$. The answer is (B).

Question A.5 [5 marks]

Suppose that $f(x) = \Pi_{i=5}^{15}(x - i)$. The value of $f'(10)$ is

(A) -12500

(B) 13750

(C) -14400

(D) 11000

Solution: Use the product rule with $u = (x - 10)$, and $v = \frac{f(x)}{x-10}$. When $x - 10$ is substituted, u vanishes; we do not need to calculate v' explicitly. We find

$$f'(10) = (10 - 10)v'(10) + u'(10)v(10) = v(10).$$

Now

$$\begin{aligned} v(10) &= (10-5)(10-6)\dots(10-9)(10-11)(10-12)\dots(10-15) \\ &= (-1)^5(5!)^2 \\ &= -14400. \end{aligned}$$

The answer is (C).

Question A.6 [5 marks]

The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

and the identity matrix is denoted by $\mathbf{I}$. The matrix exponential is defined by

$$\begin{aligned} \exp(\theta\mathbf{M}) &= \sum_{n=0}^{\infty} \frac{\theta^n}{n!}\mathbf{M}^n \\ &= \mathbf{I} + \frac{\theta}{1!}\mathbf{M} + \frac{\theta^2}{2!}\mathbf{M}^2 + \frac{\theta^3}{3!}\mathbf{M}^3 + \dots. \end{aligned}$$

Then $\exp(\theta\mathbf{M})$ is equal to

(A) $\begin{bmatrix} \cosh\theta & -i\sinh\theta \\ i\sinh\theta & \cosh\theta \end{bmatrix}$

(B) $\begin{bmatrix} \cosh\theta & -i\sinh\theta \\ -i\sinh\theta & \cosh\theta \end{bmatrix}$

(C) $\begin{bmatrix} 1 & \exp(-i\theta) \\ \exp(i\theta) & 1 \end{bmatrix}$

(D) $\begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

Solution. Note that $\mathbf{M}^2 = \mathbf{I}$. This implies that $\mathbf{M}^n = \mathbf{M}$ when n is odd and $\mathbf{M}^n = \mathbf{I}$ when n is even. Working element-wise, we see

$$\begin{aligned} [\exp(\theta\mathbf{M})]_{(1,1)} &= 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots = \cosh(\theta), \\ [\exp(\theta\mathbf{M})]_{(1,2)} &= -i\left(\frac{\theta}{1!} + \frac{\theta^3}{3!} + \dots\right) = -i\sinh(\theta), \\ [\exp(\theta\mathbf{M})]_{(2,1)} &= i\left(\frac{\theta}{1!} + \frac{\theta^3}{3!} + \dots\right) = i\sinh(\theta), \\ [\exp(\theta\mathbf{M})]_{(1,1)} &= 1 + \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots = \cosh(\theta). \end{aligned}$$

The answer is (A).

Question A.7 [8 marks]

You are given that

$$\int_0^1 3f(x)dx + \int_0^4 \frac{1}{2} f\left(\frac{x}{2}\right) dx = 5,$$
$$\int_1^2 2f(x)dx + \int_0^1 f(x)dx = 7.$$

It follows that $\int_0^{2\pi} f\left(\frac{t}{\pi}\right) dt$ equals

(A) $\frac{26\pi}{7}$

(B) $\frac{31\pi}{5}$

(C) $\frac{29\pi}{7}$

(D) $\frac{24\pi}{5}$

Solution: Note two key identities here; from integration by substitution we have $\int_0^4 \frac{1}{2} f\left(\frac{x}{2}\right) dx = \int_0^2 f(x)dx$, as well as $\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx$. Let $a = \int_0^1 f(x)dx$ and $b = \int_1^2 f(x)dx$. The equations become

$$4a + b = 5$$
$$a + 2b = 7.$$

Solving gives $a = \frac{3}{7}$, $b = \frac{23}{7}$. Integration by substitution gives $\int_0^{2\pi} f\left(\frac{t}{\pi}\right) dt = \pi(a+b) = \frac{26\pi}{7}$. The answer is (A).

Question A.8 [8 marks]

The vertices of an equilateral triangle are labelled X, Y and Z. The points X, Y and Z lie on a circle of circumference 10 units. Let P and A be the numerical values of the triangle's perimeter and area, respectively. Which one of the following is true?

(A) P^2 is rational

(B) $P < A$

(C) $\frac{P}{A} = \frac{10}{\pi}$

(D) $\frac{A}{P} = \frac{5}{4\pi}$

Solution: All vertices of the triangle lie on a circle of circumference $2\pi r = 10$, so $r = \frac{5}{\pi}$. The triangle can be split into 3 smaller, identical triangles, with an angle of 30 degrees adjacent to a side of length r and a side of length $\frac{P}{3}$. Then $A = 3 \times \frac{r}{2} \times \frac{P}{3} \sin(30) = \frac{5P}{4\pi}$. Rearranging shows the answer is (D).

Full answer list: B,B,C,B,C,A,A,D.

Part B - Open Questions [60 marks]

Question B.1 [15 marks]

Let x_n be defined for $n \geq 1$ by $x_1 = a, x_2 = 62$, and

$$x_n = 2x_{n-1} - x_{n-2} \quad (n \geq 3).$$

You are given that $x_2 = 62$ and $x_{100} = 2022$. Determine a.
Solution: For any three consecutive numbers a, b, c, we know that $c = 2b - a$. Thus $c - b = b - a$ This means that the difference between pairs of consecutive numbers is constant throughout the sequence. Let this difference be d. Then, if a is the first number from the sequence, the sequence reads $a, a+d, a+2d, \ldots$. Then we have

$$x_2 = a + d = 62$$
$$x_{100} = a + 99d = 2022.$$

We solve to find $d = 20$ and $a = 42$, which is as required.

Question B.2 [15 marks]

Suppose a three digit number (none of the digits are zero) x is chosen, and underneath it are written all the permutations of its digits. All of these numbers are added, giving $S(x)$. For example, $S(112) = 444$. What is the largest value x can take such that $S(x) = 1221$?
Solution: Begin with numbers of the form abc, with a, b, c distinct. Considering the sum gives

$$S(x) = 100(a+b+c) + 10(a+b+c) + 2(a+b+c) \neq 1221,$$

by considering divisibility by 2. If we choose the form aaa, then there are no permuations and $S(x)$ is bounded above by $999 < 1221$. The only possible solution is of the form aab with $a \neq b$. In this case we require

$$S(x) = 100(a+a+b) + 10(a+a+b) + a + a + b = 222a + 111b = 1221.$$

If $b = 9$, then the only solution is $a = 1$. Thus 911 will work here. Any larger solution would require $b = 9$, but this forces $a = 1$ so there is no larger answer possible. The answer is 911.

Question B.3 [15 marks]

For each non-negative integer n, define the function $f_n(x)$ by

$$f_n(x) = \sum_{i=0}^{n} \frac{x^i}{i!}.$$

a) Show that $f_n'(x) = f_{n-1}(x)$ for $n \geq 1$.

b) Suppose $a \in \mathbb{R}$ with $f_n(a) = 0$. Show that $a < 0$.

c) Let a, b (with $a < b$) be distinct real roots of $f_n(x) = 0$ for some $n \geq 2$. Show that $f_n'(a) f_n'(b) > 0$. By a sketch or otherwise, explain why there must be $c \in (a, b)$ such that $f_n(c) = 0$.

d) Deduce that $f_n(x) = 0$ has at most one real root. How many real roots does $f_n(x) = 0$ have? *Hint: Consider n odd and n even separately.*
Solution:
a) Differentiation gives, for $n \geq 1$

$$f_n'(x) = \sum_{i=0}^{n} \frac{i x^{i-1}}{i!} = \sum_{i=1}^{n} \frac{x^{i-1}}{(i-1)!} = \sum_{i=0}^{n-1} \frac{x^i}{i!} = f_{n-1}(x).$$

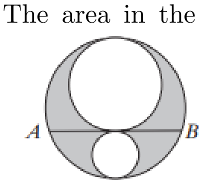
b) Assume $a \geq 0$. Then every term of $f_n(a)$ is positive and so $f_n(a) > 0$. Any root of f_n must be negative.
c) We have

$$f_n'(a) = f_n(a) - \frac{a^n}{n!} = \frac{a^n}{n!}$$
$$f_n'(b) = f_n(b) - \frac{b^n}{n!} = \frac{b^n}{n!},$$

which both have the same sign since $a, b < 0$. We conclude $f_n'(a) f_n'(b) > 0$. Then a sketch quickly shows a root $c \in (a, b)$, since the sign of f_n at a and b is the same.
d) Assume toward a contradiction that $f_n(x)$ has 2 or more roots. Let the smallest two roots be x_1 and x_2. Then by (c), there is another root between x_1 and x_2, which contradicts the minimality of x_2. We find that $f_n(x) = 0$ has at most one root. When n is odd, $f_n(-\infty) = -\infty$ and $f_n(\infty) = \infty$; the intermediate value theorem implies one real root. When n is even, $f_n(x) = 0$ has no real roots.

Question B.4 [15 marks]

The area in the shaded region of the diagram is 2π. Find the length AB.



Solution: Let C be the center of the larger inner circle. Let D be the center of the smaller inner circle. Let C have radius a, D have radius b. Let O be the center of the outer circle, whilst letting M be the point of intersection of the two inner circles. Then the shaded area is $\pi(a+b)^2 - \pi a^2 - \pi b^2 = 2\pi ab = 2\pi$, so $ab = 1$. Consider the (right-angled) triangle AMO. We have OA $= a + b$, whilst

OM $= 2a - (a+b) = a - b$. Pythagoras' theorem then yields, with $x =$ AM, that

$$x^2 = (a+b)^2 - (a-b)^2 = 4ab.$$

We find that $x = 2ab = 2$, which is half of the the length of the chord. The answer is 4.