SIAM Maths Challenge Questions

Part A - Multiple-choice Questions [40 marks]

In this section, full marks are awarded for selection of the correct option; there are no method marks.

Question A.1 [3 marks]

Define an *unfortunate number* as a number which is 13 times the sum of its digits. How many unfortunate numbers are there?

(A) None

(B) 3

(C) 5

(D) Infinitely many

Question A.2 [3 marks]

Let $q \geq 0$. The minimum value of

$$I(q) = \int_0^1 (x-q)^2 dx + \int_{-q}^{q} \sin^9(x)(|x| + \cos(x) + q)dx$$

as q varies is achieved at

(A) $q = 0$

(B) $q = \frac{1}{2}$

(C) $q = 1$

(D) Multiple values of q

Question A.3 [3 marks]

How many integers n are there such that $\frac{n}{100-n}$ is also an integer?

(A) 12

(B) 15

(C) 18

(D) 21

Question A.4 [5 marks]

Assume angles are given in degrees. The sum

$$\cos^2(1) + \cos^2(2) + \cdots + \cos^2(270)$$

equals

(A) 134

(B) 134.5

(C) 135

(D) 135.5

Question A.5 [5 marks]

Suppose that $f(x) = \Pi_{i=5}^{15}(x - i)$. The value of $f'(10)$ is

(A) -12500

(B) 13750

(C) -14400

(D) 11000

Question A.6 [5 marks]

The matrix $\mathbf{M}$ is given by

$$\mathbf{M} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

and the identity matrix is denoted by $\mathbf{I}$. The matrix exponential is defined by

$$\begin{aligned} \exp(\theta\mathbf{M}) &= \sum_{n=0}^{\infty} \frac{\theta^n}{n!}\mathbf{M}^n \\ &= \mathbf{I} + \frac{\theta}{1!}\mathbf{M} + \frac{\theta^2}{2!}\mathbf{M}^2 + \frac{\theta^3}{3!}\mathbf{M}^3 + \ldots. \end{aligned}$$

Then $\exp(\theta\mathbf{M})$ is equal to

(A) $\begin{bmatrix} \cosh\theta & -i\sinh\theta \\ i\sinh\theta & \cosh\theta \end{bmatrix}$

(B) $\begin{bmatrix} \cosh\theta & -i\sinh\theta \\ -i\sinh\theta & \cosh\theta \end{bmatrix}$

(C) $\begin{bmatrix} 1 & \exp(-i\theta) \\ \exp(i\theta) & 1 \end{bmatrix}$

(D) $\begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

Question A.7 [8 marks]

You are given that

$$\int_0^1 3f(x)dx + \int_0^4 \frac{1}{2} f\left(\frac{x}{2}\right) dx = 5,$$
$$\int_1^2 2f(x)dx + \int_0^1 f(x)dx = 7.$$

It follows that $\int_0^{2\pi} f\left(\frac{t}{\pi}\right) dt$ equals

(A) $\frac{26\pi}{7}$

(B) $\frac{31\pi}{5}$

(C) $\frac{29\pi}{7}$

(D) $\frac{24\pi}{5}$

Question A.8 [8 marks]

The vertices of an equilateral triangle are labelled X, Y and Z. The points X, Y and Z lie on a circle of circumference 10 units. Let P and A be the numerical values of the triangle's perimeter and area, respectively. Which one of the following is true?

(A) P^2 is rational

(B) $P < A$

(C) $\frac{P}{A} = \frac{10}{\pi}$

(D) $\frac{A}{P} = \frac{5}{4\pi}$

Part B - Open Questions [60 marks]

To get full marks for each question, your workings (showing the methods you have used) are just as important as the final answer. Any answers to the questions in part B which are obtained by guessing, trial and error, or by means of code will be awarded low/partial credit. In contrast, part of the marks may already be earned by writing down some correct observations, or handing in a semi-successful attempt, so we do encourage you to try to solve these problems!

Question B.1 [15 marks]

Let x_n be defined for $n \geq 1$ by $x_1 = a, x_2 = 62$, and

$$x_n = 2x_{n-1} - x_{n-2} \qquad (n \geq 3).$$

You are given that $x_{100} = 2022$. Determine a.

Question B.2 [15 marks]

Suppose a three digit number (none of the digits are zero) x is chosen, and underneath it are written all the permutations of its digits. All of these numbers are added, giving $S(x)$. For example, $S(112) = 444$. What is the largest value x can take such that $S(x) = 1221$?

Question B.3 [15 marks]

For each non-negative integer n, define the function $f_n(x)$ by

$$f_n(x) = \sum_{i=0}^{n} \frac{x^i}{i!}.$$

a) Show that $f_n'(x) = f_{n-1}(x)$ for $n \geq 1$.

b) Suppose $a \in \mathbb{R}$ with $f_n(a) = 0$. Show that $a < 0$.

c) Let a, b (with $a < b$) be distinct real roots of $f_n(x) = 0$ for some $n \geq 2$. Show that $f_n'(a)f_n'(b) > 0$. By a sketch or otherwise, explain why there must be $c \in (a, b)$ such that $f_n(c) = 0$.

d) Deduce that $f_n(x) = 0$ has at most one real root. How many real roots does $f_n(x) = 0$ have? *Hint: Consider n odd and n even separately.*

Question B.4 [15 marks]

The area in the shaded region of the diagram is 2π. Find the length AB.