SIAM Maths Challenge 2023 Questions

The SIAM Maths challenge 2023 questions are below, comprising of two sections (Part A and part B). You may work in teams of up to three members. These questions test logic and lateral application of familiar ideas but will not require maths knowledge beyond A-level study. The best scoring teams from each year group will win prizes!

Please submit your solutions either (a) online via email of a single, scanned pdf to siam@nottingham.ac.uk, or (b) in person to the maths building reception. **Submit solutions by 1pm**, with prizes and pizza in Pope A14 at 2pm.

Part A - Multiple-choice Questions [40 marks]

In this section, full marks are awarded for the correct answer; there are no method marks. For some questions in Section A there are options to choose from. Your answers should be clearly noted/circled for each question to ensure they are seen and can easily be marked!

Question A.1 [3 marks]

How many three-digit multiples of 9 consist only of odd digits?

(A) 3

(B) 7

(C) 11

(D) 15

Question A.2 [3 marks]

Let $q \geq 0$ and let $n > 0$ be an odd integer. Define $I(q)$ by

$$I(q) = \int_0^2 (x-q)^n dx + \int_{-q}^{q} q \sin^n(x) \cos(x) dx.$$

Then $I(q) = 0$ provided

(A) $q = 0$

(B) $q = \frac{1}{2}$

(C) $q = 1$

(D) $0 \leq q \leq 1$

Question A.3 [3 marks]

What is the smallest integer whose digits add to 2001?

Question A.4 [4 marks]

The equation below is satisfied by exactly three different (real) values of x, and the sum of the two lowest of these values is -3.

$$1 - \frac{1}{x^2} + k(x+1)^3 = 0.$$

Find the value of k.

Question A.5 [5 marks]

Suppose that $y = 2x + 3x^2 + 5x^3 + \ldots$, so that the coefficient of x^n is the n^{th} prime number. The value of $\frac{d^2y}{dx^2} + \frac{d^5y}{dx^5}$ at $x = 0$ is

(A) 1126

(B) 1226

(C) 1326

(D) 1426

Question A.6 [6 marks]

Suppose a fair six-sided die is rolled six times. What is the probability of the sum of these rolls being divisible by 6?

Question A.7 [8 marks]

Given that there are positive real numbers a, b, c that satisfy

$$\int_a^b \log_c(\sin^4 x \tan^2 x)dx = 1,$$
$$\int_a^b \log_c(\sin^2 x \cos^2 x)dx = 3,$$

it follows that $\int_a^b \log_c(\sin^4 x \cos^2 x)$ equals

(A) 4

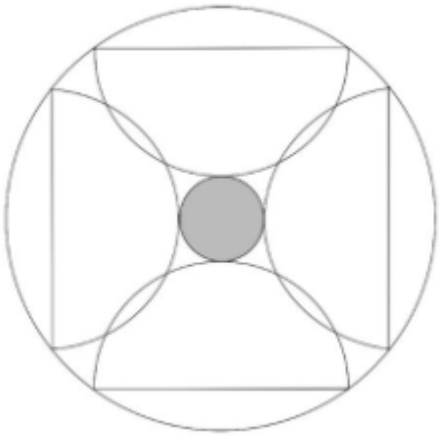
(B) 5

(C) 6

(D) 7

Question A.8 [8 marks]

The inner circle has area 4. Each semicircle has area 18. What is the area of the outer circle?



Part B - Open Questions [60 marks]

To get full marks for each question, your workings (showing the methods you have used) are just as important as the final answer. Any answers to the questions in part B which are obtained by guessing, trial and error, or by means of code will be awarded low/partial credit. In contrast, part of the marks may already be earned by writing down some correct observations, or handing in a semi-successful attempt, so we do encourage you to try to solve these problems!

Question B.1 [15 marks]

You are given that there is a unique real value α which satisfies

$$\alpha^3 + \alpha^2 = 1.$$

(i) Show that $\alpha \in [0, 1]$
(ii) Show that $\alpha^4 = -1 + \alpha + \alpha^2$

(iii) Four functions of α are given in (a) to (d) below. In a similar manner to part (ii), each is equal to a quadratic expression

$$A + B\alpha + C\alpha^2$$

in α where A, B, C are integers. (So in (ii) we found $A = -1$, $B = 1$, $C = 1$.) You may assume in each case that the quadratic expression is unique. In each case below find the quadratic expression in α.

(a) α^{-1}

(b) $1 - \alpha + \alpha^2 - \alpha^3 + \ldots$

(c) $(1 - \alpha)^{-1}$

(d) $(1 + \alpha)(1 + \alpha^2)(1 + \alpha^4)(1 + \alpha^8)\ldots$

Question B.2 [15 marks]

Given that $34! = 295232799cd960414084761860964 35ab000000$, find a, b, c, and d.

Question B.3 [15 marks]

Consider the following rules. Begin with a word of any length. You may then, in any order

- Add any consonant to the end of the word
- Add any vowel to the start of the word
- Delete any two consecutive consonants or two consecutive vowels (these need not be the same letter - BC may be deleted together, for example)
- Double the word (i.e. ABC $\rightarrow$ ABCABC)

These rules can be applied as often as you want and in any order.

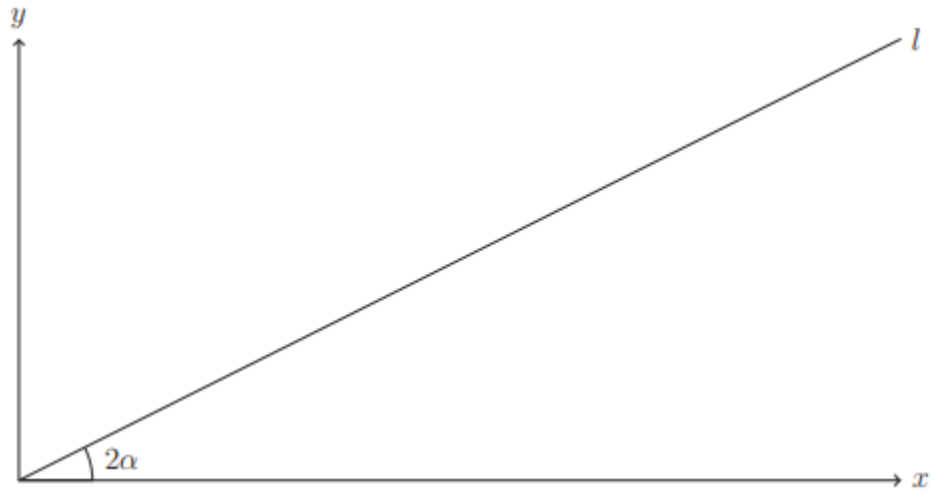
(a) Starting from LEAD, make GOLD

(b) Prove that all words can become the empty word (with no letters)

(c) Can any word be made from the empty word? Justify your answer

Question B.4 [15 marks]

The line l passes through the origin at angle 2α above the x-axis, where $2\alpha < \frac{\pi}{2}$.



Circles C_1 of radius one and C_2 of radius three are drawn between l and the x-axis, just touching both lines.

(a) What is the centre of circle C_1?
(b) What is the equation of circle C_1?
(c) For what value of α do circles C_1 and C_2 touch?
(d) For this value of α (for which the circles C_1 and C_2 touch) a third circle, C_3, larger than C_2, is to be drawn between l and the x-axis. C_3 just touches both lines and also touches C_2. What is the radius of this circle C_3?
(e) For the same value of α, what is the area of the region bounded by the x-axis and the circles C_1 and C_2?